Problems 1 – 3

Let us choose $|\ell, m\rangle$ as the common eigenvector for angular momentum operators $L^2$ and $L_z = L_3$ such that

\[ L_z |\ell, m\rangle = \hbar m |\ell, m\rangle, \quad \text{and} \]
\[ L^2 |\ell, m\rangle = \hbar^2 (\ell + 1) |\ell, m\rangle \]

where $m$ is a quantum number $-\ell \leq m \leq \ell$. The eigenvectors $|\ell, m\rangle$ form a complete set of orthonormal basis with $\langle \ell', m' | \ell, m \rangle = \delta_{\ell', \ell} \delta_{m', m}$.

In addition, the lowering and raising operators are defined as

\[ L_- \equiv L_1 - iL_2 = L_x - iL_y \]
\[ L_+ \equiv L_1 + iL_2 = L_x + iL_y. \]

Problem (1)

Find the coefficients $c_m$ and $d_m$ for

\[ L_- |\ell, m\rangle = c_m |\ell, m - 1\rangle \]
\[ L_+ |\ell, m\rangle = d_m |\ell, m + 1\rangle \]

where $L^2 = L_1L_i - L_2^2 + L_3^2$.

Problem (2)

The angular momentum operators have interesting relations

\[ L_- |\ell, m\rangle = \hbar \sqrt{\ell(\ell + 1) - m(m - 1)} |\ell, m - 1\rangle, \]
\[ L_+ |\ell, m\rangle = \hbar \sqrt{\ell(\ell + 1) - m(m + 1)} |\ell, m + 1\rangle. \]

Calculate $\langle L_x \rangle$, $\langle L_y \rangle$, $\langle L_z^2 \rangle$ and $\langle L_y^2 \rangle$ in such a state denoted by $|\ell, m\rangle$.

Problem (3)

The eigenvalue of $L_z$ has a maximum value $m_{\text{MAX}} = \ell$ such that $L_+ |\ell, \ell\rangle = 0$. In the spherical coordinate basis, we have

\[ L_z = -i\hbar \frac{\partial}{\partial \phi}, \quad \text{and} \]
\[ L_+ = \hbar e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right). \]
Let us define
\[ U_{\ell,m} \equiv \langle r, \theta, \phi | \ell, m \rangle \quad \text{and} \quad U_{\ell,\ell}(r, \theta, \phi) = R_{\ell,\ell}(r) \Theta_{\ell,\ell}(\theta) \Phi_{\ell}(\phi). \]

Apply separation of variables and show that
\[ \Phi_{\ell}(\phi) = A_{\phi} e^{i\ell \phi}, \]
\[ \Theta_{\ell,\ell}(\theta) = A_\theta (\sin \theta)^\ell, \]
and
\[ U_{\ell,\ell}(r, \theta, \phi) = AR_{\ell,\ell}(\sin \theta)^\ell e^{i\ell \phi} \]
where \( A_{\phi}, A_\theta \) and \( A \) are normalization constants.

**Problems 4 and 5**

For the hydrogen atom, the ground state wave function is
\[ \psi(r) = \psi_{100}(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \]
and the wave functions of the hydrogen atom with \( n = 2, \ell = 1, m = \pm 1 \) are
\[ \psi_{21\pm 1}(\vec{r}) = \mp \frac{1}{\sqrt{\pi a^2}} \frac{1}{8a^2} r e^{-r/2a} \sin \theta e^{\pm i \phi}. \]

**Problem (4) [Griffiths 4.13]**

(a) Find \( \langle R \rangle \) and \( \langle R^2 \rangle \) for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr radius.

(b) Find \( \langle X \rangle \) and \( \langle X^2 \rangle \) for an electron in the ground state of hydrogen atom. *Hint:* This requires no new integration–note that \( R^2 = X^2 + Y^2 + Z^2 \), and exploit the symmetry of the ground state.

(c) Find \( \langle X^2 \rangle \) in the state \( n = 2, \ell = 1, m = 1 \) with \( X = R \sin \theta \cos \phi \).

**Problem (5) [Griffiths 4.15]**

A hydrogen atom starts out in the following linear combination of the stationary states \( n = 2, \ell = 1, m = 1 \) and \( n = 2, \ell = 1, m = -1 \)
\[ \Psi(\vec{r}, 0) = \frac{1}{\sqrt{2}} (\psi_{211} + \psi_{21} - 1). \]

(a) Construct \( \Psi(\vec{r}, t) \), Simplify it as much as you can.

(b) Find the expectation value of the potential energy, \( \langle V \rangle \). Give both the formula and the actual number in electron volts.