Problem (1)

Let us consider the following operators on a Hilbert space $V^3$:

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

(i) Consider the state

$$|\psi\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$$

in the $L_z$ basis. If $L_z$ is measured in this state and a result +1 is obtained, what is the state after the measurement? How probable was this result? If $L_z$ is measured, what are the outcomes and respective probabilities?

(ii) A particle is in a state for which the probabilities are

$$P(\ell_z = 1) = 1/4,$$
$$P(\ell_z = 0) = 1/2,$$ and
$$P(\ell_z = -1) = 1/4.$$

The most general normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2} |\ell_z = 1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}} |\ell_z = 0\rangle + \frac{e^{i\delta_3}}{2} |\ell_z = -1\rangle$$

If $|\psi\rangle$ is a normalized state then the state $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors $e^{i\delta_i}$ are irrelevant? Calculate $P(\ell_x = 1)$, $P(\ell_x = 0)$, and $P(\ell_x = -1)$, for

(a) $\delta_1 = \delta_2 = \delta_3 = 0,$
(b) $\delta_1 = \delta_3 = 0,$ and $\delta_2 = \pi,$
(c) $|\psi'\rangle = e^{-i\delta_1} |\psi\rangle.$
Problem (2)

For a free particle in one dimension, the Schrödinger Equation is

\[ i\hbar \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle \quad \text{with} \quad H = \frac{P^2}{2m}. \]

(a) Show that the wave function \( \phi(p) \) in the momentum space or the \( p \)–basis is

\[ \phi(p) = \langle p | \Psi \rangle = Ne^{-\frac{i}{\hbar} \left( \frac{p^2}{2m} \right)t} \]

where \( N \) is the normalization constant.

(b) Find the wave function \( \Psi(x,t) = \langle x | \Psi \rangle \) in the coordinate space or the \( x \)–basis by applying the Fourier transform

\[ \Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \phi(\hbar k) \, dk \]

where \( \phi(p) = \phi(\hbar k) \) with \( k = p/\hbar \).

Problem (3)

For a particle moving in an infinite square well of width \( 2a \) with the potential energy

\[ V(x) = \begin{cases} 
0, & \text{for } -a \leq x \leq a \text{ with } a > 0, \\
\infty, & \text{otherwise},
\end{cases} \]

its normalized wave function inside the well at time \( t = 0 \) is

\[ \Psi(x,0) = C\left[ \sin \frac{\pi x}{a} + \frac{1}{4} \cos \frac{3\pi x}{2a} \right] \]

and \( \Psi(x,0) = 0 \) for \( x^2 \geq a^2 \).

(a) Calculate the coefficient \( C \).

(b) What is the wave function \( \Psi(x,t) \)?

(c) If a measurement of total energy is made, what are the possible results of such a measurement, and what is the probability to measure each of them?

(d) What is the expectation value of the energy \( \langle E \rangle \)?
Problem (4)

Let us consider a one-dimensional quantum harmonic oscillator with the Hamiltonian

\[ H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2, \quad \text{and} \]

\[ H|n\rangle = E_n|n\rangle. \]

(a) Find \( \langle X \rangle, \langle P \rangle, \langle X^2 \rangle, \langle P^2 \rangle \) and \( \Delta X \Delta P \) in the state \( |n\rangle \).

(b) What is the uncertainty relation for the ground state.