Problem (1)

In the energy basis of a harmonic oscillator, we have

\[ X = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \quad \text{and} \quad P = -i\sqrt{\frac{\hbar m\omega}{2}} (a - a^\dagger) \]

where \( a \) and \( a^\dagger \) are the lowering and raising operators, respectively.

Calculate the following matrix elements for a quantum oscillator

(a) \( \langle 2 | X^3 | 0 \rangle \), and
(b) \( \langle 2 | X^3 | 1 \rangle \).

Problem (2)

Apply the generating function for the Hermite polynomial

\[ G(\xi, s) = e^{\xi^2 - (s-\xi)^2} = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi) \]

and show that

(a) \( H_n(\xi) = \frac{\partial^n G(\xi, s)}{\partial s^n} \big|_{s=0} \);
(b) \( \frac{\partial^n G(\xi, s)}{\partial s^n} \big|_{s=0} = (-1)^n e^{\xi^2} \frac{\partial^n}{\partial \xi^n} e^{-\xi^2} = H_n(\xi) \); and
(c) \( \int_{-\infty}^{\infty} d\xi H_n(\xi) H_m(\xi) e^{-\xi^2} = \begin{cases} 0 & \text{if } n \neq m, \\ \sqrt{\pi} 2^n \cdot n! & \text{for } n = m. \end{cases} \)

(d) Find the normalized wave function \( u_n(x) \) with

\[ u_n(\xi) = A_n H_n(\xi) e^{-\xi^2}, \]

where \( A_n \) is real, \( \xi = \alpha x \), \( \alpha = \sqrt{m\omega/\hbar} \), and \( u_n(x) \) is an eigenfunction for the one-dimensional Simple Harmonic Oscillator, corresponding to the energy \((n + 1/2)\hbar\omega\).
Problem (3)

A simple harmonic oscillator has the Hamiltonian expressed as

\[ H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2X^2. \]

(a) Show that

\[ \Psi(x, t) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left[ -\frac{m\omega}{2\hbar} \left( x^2 + \frac{\alpha^2}{2} \left( 1 + e^{-2i\omega t} \right) + \frac{iht}{m} - 2\alpha xe^{-i\omega t} \right) \right] \]

satisfies the Schrödinger equation for the harmonic oscillator. Here \( \alpha \) is a real constant with the dimensions of length.

(b) Compute \( \langle X \rangle \) and \( \langle P \rangle \) and check that Ehrenfest’s theorem is satisfied.

Problem (4)

A particle moves in three dimensions in a potential of the form

\[ V(\vec{r}) = \begin{cases} \frac{1}{2}m\omega^2(x^2 + y^2), & \text{for } -a \leq z \leq a \text{ with } a > 0, \text{ and} \\ \infty, & \text{for } z^2 > a^2, \end{cases} \]

where \( m \) and \( \omega \) are constants.

Let us consider the wave function of this system as

\[ \psi_{l,m,n}(\vec{r}) = \psi_{a}^{l}(x)\psi_{b}^{m}(y)\psi_{c}^{n}(z). \]

Find formulas for its energy eigenvalues, the degeneracy of each energy level, and the eigenfunctions associated with them.